

EXERCISE IV

Castelnuovo's Bound

This exercise is about the following theorem valid for an algebraic curve X and a holomorphic embedding $\phi : X \rightarrow \mathbb{C}P^n$ with non-degenerate image (see Definition 5.4.8). For d the degree of $\phi(X)$ and g the genus of X we write

$$d = 1 + m(n - 1) + \epsilon \quad \text{with} \quad m \geq 1 \quad \text{and} \quad 0 \leq \epsilon \leq n - 1.$$

Then, the relation

$$g \leq \frac{1}{2}(n - 1)(m - 1)m + m\epsilon$$

holds and is called *Castelnuovo's bound* on the genus of X .

1. Proof of the bound. We take the following statement as given (it is a consequence of the *General Position Lemma*): for D a hyperplane divisor on $\phi(X) \subset \mathbb{C}P^n$ and $k \in \mathbb{Z}$,

$$\dim L(kD) - \dim L((k - 1)D) \geq \begin{cases} 1 + k(n - 1) & \text{for } 1 + k(n - 1) \leq d \\ d & \text{for } 1 + k(n - 1) \geq d. \end{cases} \quad (1)$$

- (a) Prove that for $k \geq 1$

$$\dim L(kD) \geq 1 + m + \frac{1}{2}(n - 1)(m + 1)m + d(k - m)$$

by expanding $\dim L(kD)$ in a telescoping sum over the differences (1).

- (b) Combine (a) with the Riemann-Roch-Theorem to deduce Castelnuovo's bound.

2. Examples.

- (a) Show that the genus of every smooth projective plane curve $X \subset \mathbb{C}P^2$, $\phi = \text{id}$, achieves Castelnuovo's bound.
- (b) Suppose X is an algebraic curve of genus g , and D is a divisor on X of degree $d \geq 2g + 1$. Recall that D is very ample. Consider the associated holomorphic embedding $\phi_D : X \rightarrow \mathbb{C}P^n$, determine n and show that the genus of X achieves Castelnuovo's bound.

- (c) Consider a non-hyperelliptic curve of genus $g \geq 3$ and its canonical map ϕ_K (see Definition 7.2.3). Show that the genus of X achieves Castelnuovo's bound.
- (d) Show that for all $d \geq n \geq 2$, there exist holomorphically embedded Riemann surfaces $X \subset \mathbb{C}P^n$ whose genera achieve Castelnuovo's bound.
- (e) Assume that $X \subset \mathbb{C}P^3$ is a holomorphically embedded Riemann surface of degree d whose genus achieves Castelnuovo's bound. Show that its genus is

$$g = \begin{cases} (\frac{1}{2}d - 1)^2 & \text{if } d \text{ is even} \\ \frac{1}{2}(d - 1)(d - 3) & \text{if } d \text{ is odd.} \end{cases}$$

Show that if $d \geq 7$, X lies on a (unique) quadric.