Math 255: Algebraic Curves
UCB, Spring 2009
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Exercise II

Classification of Complex Tori

For L a lattice in $\mathbb{C}$, we denote by $X_L := \mathbb{C}/L$ be the associated complex torus, and by $\pi_L : \mathbb{C} \longrightarrow X_L$ the quotient map. Recall that X_L is an abelian group.

1. Let L and L' be lattices in $\mathbb{C}$. Show that for every holomorphic map $F : X_L \longrightarrow X_{L'}$ there exists a linear map $G : \mathbb{C} \longrightarrow \mathbb{C} : z \longmapsto \alpha z + \beta$, with constants $\alpha, \beta \in \mathbb{C}$, such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C} \\ {\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi_{L'}} \\ X_L & \xrightarrow[F]{} & X_{L'} \end{array}$$

   is commutative. Show that α is uniquely determined and that $\alpha L \subset L'$.

   Hint: Show the linearity by proving that G' is constant.

2. Show that F is a group homomorphism if and only if $F(0) = 0$.

3. Show that the degree of F is the index of αL in L', where α is the unique constant from 1 .

4. Suppose $F : X_L \longrightarrow X_L$ is a group isomorphism, and α is the unique constant from 1 . Show that either

   (a) L is a square lattice and α is a 4th root of unity;
   (b) L is hexagonal and α is a 6th root of unity; or
   (c) L is neither square nor hexagonal and $\alpha = \pm 1$.

   In particular, the group of group automorphisms of a complex torus has either four, six or two elements.

5. Suppose $\tau, \tau' \in \mathbb{C}$ with non-negative imaginary parts, and let $L_\tau, L_{\tau'}$ be the associated lattices. Using 3., prove that the tori X_{L_τ} and $X_{L_{\tau'}}$ are isomorphic if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

   such that

$$\tau = \frac{a\tau' + b}{c\tau' + d} .$$