Math 255: Algebraic Curves
UCB, Spring 2009
Dr. Konrad Waldorf

Exercise I

Meromorphic Functions on Complex Tori

Let L be a lattice in $\mathbb{C}$, and let $X := \mathbb{C}/L$ be the associated complex torus. Recall that by Remark 2.1.7 iii), meromorphic functions on X are in bijection to L-periodic meromorphic functions on $\mathbb{C}$ (= 'elliptic functions').

1. Show that every complex torus $\mathbb{C}/L$ is isomorphic to a torus $\mathbb{C}/L_\tau$ with $L_\tau := \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$.

2. In the following, $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Show that the *theta-function*

$$\theta(z) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n^2\tau + 2nz)}$$

   is analytic on $\mathbb{C}$ and satisfies

$$\theta(z+\tau) = e^{-\pi\mathrm{i}(\tau+2z)}\theta(z).$$

   Show that the zeroes of θ are $\frac{1+\tau}{2} + m + n\tau$, for $n, n \in \mathbb{Z}$, and that they are all simple (Hint: integrate θ'/θ around the contour of a fundamental domain).

3. For any $x \in \mathbb{C}$, let

$$\theta^x(z) := \theta\left(z - \frac{1+\tau}{2} - x\right).$$

   be the *translated theta-function*. For complex numbers $x_1, ..., x_m$ and $y_1, ..., y_n$, consider the meromorphic function

$$R(z) := \frac{\prod_{i=1}^m \theta^{x_i}(z)}{\prod_{i=1}^n \theta^{y_i}(z)} \tag{1}$$

   on $\mathbb{C}$. Show that R is L_τ-periodic if and only if $m = n$ and

$$\sum_{i=1}^m x_i - \sum_{i=1}^n y_i \in \mathbb{Z}. \tag{2}$$

   Compute the zeroes and poles of R.

4. Show that any meromorphic function on $X = \mathbb{C}/L_\tau$ is of the form (1):

(a) Let f be any meromorphic function on X, let $p_1, ..., p_m$ be its zeroes and $q_1, ..., q_n$ its poles, with repetitions for zeroes and poles of higher order. Show that $m = n$.

(b) Recall that X is an abelian group. Show that

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{n} q_i. \tag{3}$$

Hints:

i. Assume $p_0, q_0 \in \mathbb{C}$ with $p_0 \neq q_0$, such that (3) (with the sum starting at 0) is true.

ii. Consider the function R from (1) associated to the sequences $p_0, p_1, ..., p_m$ and $q_0, q_1, ..., q_n$. Compute the zeroes and poles of the meromorphic function $g := R/f$.

iii. Show that the holomorphic map $G : X \longrightarrow \mathbb{C}_\infty$ which is associated to g is an isomorphism, which is a contradiction since X and $\mathbb{C}_\infty$ have different topological genera.

(c) Show that there exist complex numbers $x_1, ..., x_m$ and $y_1, ..., y_m$ with $\pi(x_i) = p_i$ and $\pi(y_i) = q_i$ satisfying (2). Conclude that the function R associated to the x_i and y_i is equal to cf, for a constant c.